10:35

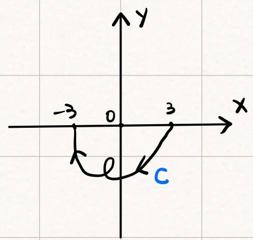
1. Show that if $w(t) = u(t) + i v(t)$ is continuous on $a \le t \le b$,

then (a) $\int_{-b}^{-a} w(-\tau)\, d\tau = \int_a^b w(t)\, dt$

(b) $\int_a^b w(t)\, dt = \int_\alpha^\beta w[\phi(\tau)]\, \phi'(\tau)\, d\tau$.

where $t = \phi(\tau)$, $\alpha \le \tau \le \beta$, $\phi'(\tau) > 0$ continuous.

2. Use antiderivative to evaluate the integral



$z^{\frac{1}{2}} = \exp(\frac{1}{2} \log z)$.

$$\int_C z^{\frac{1}{2}} dz, \quad (|z| > 0,\ 0 < \arg z < 2\pi).$$

P. 131. 132.

$\int f(z(t))\, z'(t)\, dt$. works

3. Let R be the region bounded by a simple closed contour C, prove that

$$A = \frac{1}{2i} \oint_C \bar{z}\, dz, \quad \text{where } A \text{ is the area of } R.$$

Review the definition of contour

$z'(t) = 1+i$

$z(t) = t+it.$

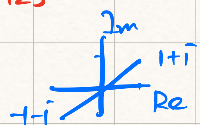
$t \in [-1,1]$

Given an arc $z(t)$, $a \le t \le b$,

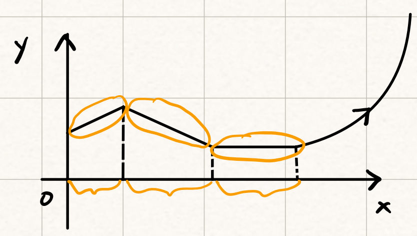
(i) $z(t)$ is smooth if $z'(t)$ is continuous on $[a,b]$ and non-zero on (a,b). P. 123

$z(t) = t^3 + it^3$, $-1 \le t \le 1$

$z'(t) = 3t^2 + i \cdot 3t^2$



(ii) $z(t)$ is a contour if $z(t)$ is a piecewise smooth arc.



(iii) A contour $z(t)$ is simple if it does not cross itself.

(iv) A contour $z(t)$ is closed if it is simple except for $z(b) = z(a)$

└ A closed contour is positively oriented if it has counterclockwise direction.

Example:

~~simple~~ ○	simple ✓	simple ○	simple ✓
~~closed~~ ○	closed ○	closed ✓	closed ✓

Green's theorem:

Let C be a positively oriented closed contour in a plane, let D be the region bounded by C.

$$\vec{F}(x,y) = (L(x,y),\ M(x,y))$$

$$\vec{r}(t) = (x(t),\ y(t)), \quad a \le t \le b$$

where L, M have continuous partial derivatives.

then

$$\oint_C \vec{F}\, d\vec{r} = \oint_C (L\,dx + M\,dy) = \iint_D \underbrace{\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)}_{\text{curl } f} \underbrace{dx\,dy}_{dA}.$$

"Path Independence" property

Suppose $f(z)$ is continuous in a domain D. TFAE:

(a) $f(z)$ has an antiderivative $F(z)$ throughout D.

(b) $\int_C f(z) = 0$ for all closed contour $C \subset D$.

(c) $\int_\gamma f(z)\,dz$ is path independent for $\gamma \subset D$.

way to check existence of anti-derivative.

If f is analytic in a simply connected open domain D, then f has an antiderivative in D (a sufficient condition).

1. $\int_{-b}^{-a} w(-\tau)\,d\tau = \int_{-b}^{-a} u(-\tau)\,d\tau + i\int_{-b}^{-a} v(-\tau)\,d\tau$

$$\overset{t=-\tau}{\underset{-dt=d\tau}{=}} \int_b^a u(t)(-1)\,dt + i\int_b^a v(t)(-1)\,dt.$$

$$= -\int_b^a (u(t)+iv(\tau))\,dt$$

$$= \int_a^b w(\tau)\,dt.$$

$$\int_a^b w(t)\,dt \overset{\text{want}}{=} \int_\alpha^\beta w(\phi(\tau))\,\phi'(\tau)\,d\tau.$$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

$$\underset{\alpha\le\tau\le\beta}{\overset{t=\phi(\tau)}{=}} \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} u(\phi(\tau))\phi'(\tau)\,d\tau + i\int_{\phi^{-1}(a)}^{\phi^{-1}(b)} v(\phi(t))\,\phi'(\tau)\,d\tau$$

$\phi'(\tau) > 0$

$\Rightarrow dt = \phi'(\tau)\,d\tau.$

$\phi(\alpha) = a$

$\phi(\beta) = b$

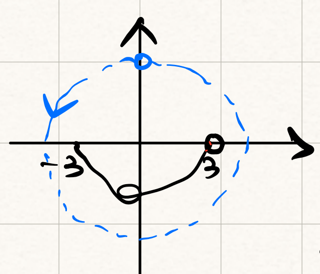
$$= \int_\alpha^\beta \left[u(\phi(\tau)) + iv(\phi(\tau))\right]\phi'(\tau)\,d\tau$$

$$= \int_\alpha^\beta w(\phi(\tau))\,\phi'(\tau)\,d\tau.$$

2. $f(z) = z^{\frac{1}{2}}$ is continuous for $0 < \arg z < 2\pi$.

but not for $z = 3$ / $\arg z = 2\pi$.

(branch cut).



Choose another branch of $z^{\frac{1}{2}}$.

$\hat{f}(z) = z^{\frac{1}{2}}$ $\left(|z|>0,\ \frac{\pi}{2} < \arg z < \frac{5\pi}{2}\right)$

$\hat{f}(z)$ coincides with $f(z)$ at $z = -3$.

and all points on C. except at $z=3$

but a point does not affect the value of the integral.

$\hat{f}$ is continuous on C.

$$\Rightarrow \int_C z^{\frac{1}{2}} dz = \int_3^{-3} \tilde{f}(z)\, dz = \tilde{F}(z)\Big|_3^{-3}$$

$$= \left[\frac{2}{3} z^{\frac{3}{2}}\right]_3^{-3} \qquad \frac{\pi}{2} < \arg z < \frac{5\pi}{2}.$$

$$= \frac{2}{3}\left(3^{\frac{3}{2}} \cdot e^{i\frac{3\pi}{2}} - 3^{\frac{3}{2}} \cdot e^{i3\pi}\right) \qquad -3 = 3e^{i\pi}, \quad 3 = 3e^{i\cdot 2\pi}.$$

$$= 2\sqrt{3}\,(-i - (-1))$$

$$= 2\sqrt{3} - 2\sqrt{3}\, i.$$

3. $\bar{z} = x - iy$, $dz = dx + i dy$.

$$\oint_C \bar{z}\, dz = \oint_C (x - iy)(dx + i dy)$$

$$= \oint_C (x dx + y dy) + i\left[\oint_C (-y) dx + x dy\right].$$

Check conditions for Green's THM.

$\vec{F_1} = (L_1, M_1)$, $L_1(x,y) = x$, $M_1(x,y) = y$

$\vec{F_2} = (L_2, M_2)$, $L_2(x,y) = -y$, $M_2(x,y) = x$

All partial derivatives are continuous.

$$\overset{\text{Green's}}{\underset{\text{THM}}{=}} \iint_D \underbrace{0}_{\frac{\partial M_1}{\partial x}} - \underbrace{0}_{\frac{\partial L_1}{\partial y}}\, dA + i \iint_D \underbrace{1}_{\frac{\partial M_2}{\partial x}} - \underbrace{(-1)}_{\frac{\partial L_2}{\partial y}}\, dA$$

$$= 2i \iint_D dA$$

$$= 2i\, A.$$

$$\Rightarrow A = \frac{1}{2i} \oint_C \bar{z}\, dz.$$

Ex 3. P. 129.